The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2

Date: August 3, 2016

Course: EE 313 Evans

Name: Set, Solution
Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and homework solution sets.
- **Power off all cell phones**
- You may use any standalone calculator or other computing system, i.e. one that is not connected to a network.
- Please do not wear hats or headphones during the exam.
- All work should be performed on the exam itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers unless instructed otherwise.**

Problem	Point Value	Your score	Topic
1	21		Continuous-Time Signals
2	21		Continuous-Time Systems
3	21		Discrete-Time Signals
4	21		Discrete-Time Systems
5	16		Transform Connections
Total	100		

Problem 2.1 Continuous-Time Signals. *21 points.*

The unilateral Laplace transform transforms a continuous-time function $x(t)$ into a function $X(s)$ of a complex-variable s as follows:

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} \, dt$$

(a) Using the Laplace transform definition above, find the Laplace transform of the continuous-time impulse $\delta(t)$. Please include the region of convergence. *6 points.* (See Lathi Example 4.2, p. 347.)

The continuous-time impulse (Dirac delta) is defined as follows:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0) \text{ if it exists.}$$

Using the second property, a.k.a. sifting or sampling property,

$$X(s) = \int_{0^-}^{\infty} \delta(t) e^{-st}\,dt = e^{-s(0)} = 1 \text{ for all } s$$

(b) Using the Laplace transform definition above, find the Laplace transform of a causal exponential signal $\exp(-t/\tau)\,u(t)$ where τ is the time constant in units of seconds where $\tau > 0$ and $u(t)$ is the unit step function. Please include the region of convergence. *9 points.*

$$X(s) = \int_{0^-}^{\infty} e^{-\frac{t}{\tau}} u(t) e^{-st}\,dt \quad \text{where } u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

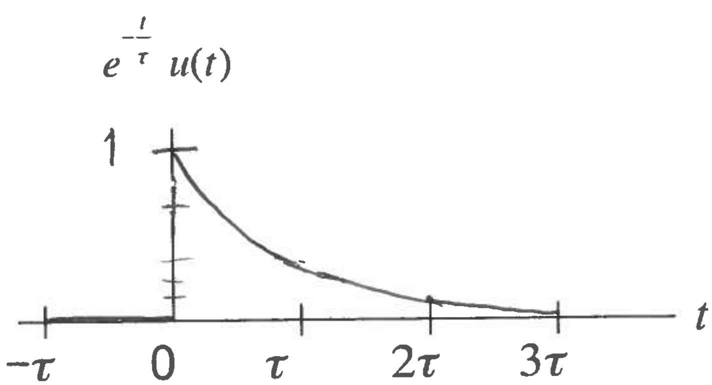
$$X(s) = \int_{0}^{\infty} e^{-(s+\frac{1}{\tau})t}\,dt = -\left.\frac{e^{-(s+\frac{1}{\tau})t}}{s+\frac{1}{\tau}}\right|_{0}^{\infty}$$

$$= \underbrace{\lim_{t\to\infty} -\frac{e^{-(s+\frac{1}{\tau})t}}{s+\frac{1}{\tau}}}_{\substack{\text{converges to 0 only if} \\ Re\{s+\frac{1}{\tau}\}>0}} + \frac{1}{s+\frac{1}{\tau}} = \frac{1}{s+\frac{1}{\tau}} \text{ for } Re\{s\} > -\frac{1}{\tau}$$

See Lathi Example 4.1 pp. 341-342.

$Re\{s+\frac{1}{\tau}\}>0 \Rightarrow Re\{s\} > -\frac{1}{\tau}$ because τ is real-valued

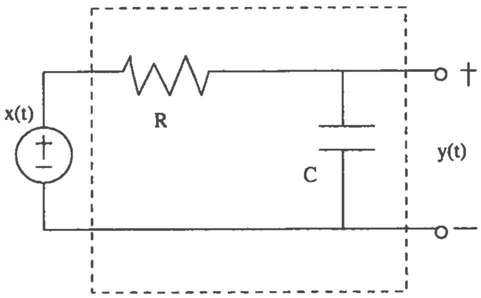
(c) Below, plot by hand the causal exponential signal $\exp(-t/\tau)\,u(t)$ where $\tau > 0$ vs. time t. *6 points.*



t	$e^{-\frac{t}{\tau}} u(t)$
0	1
τ	$e^{-1} \approx \frac{1}{3}$
2τ	$e^{-2} \approx \frac{1}{9}$
3τ	$e^{-3} \approx \frac{1}{27}$

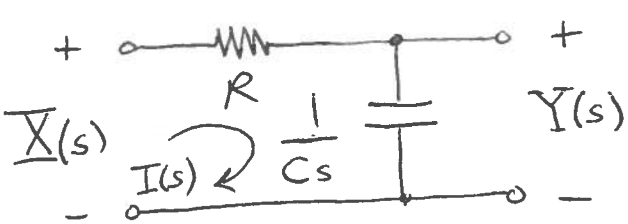
Problem 2.2 Continuous-Time Systems. *21 points.*

Consider the following analog continuous-time circuit with input $x(t)$ and output $y(t)$:



Analyze the circuit for $t > 0^-$ given that the initial voltage across the capacitor is 0 Volts.

(a) Draw the circuit in the Laplace domain. *6 points.* (See Lathi, Fig. 4.11, p. 387, and Lathi, Sec. 4.4, pp. 384-85.)



(b) Give the transfer function $H(s)$ in the Laplace domain. *6 points.*

Voltage analysis: $Y(s) = \frac{1}{Cs} I(s) = \frac{1}{Cs} \cdot \frac{X(s)}{R + \frac{1}{Cs}}$

Current analysis: $I(s) = \frac{X(s)}{R + \frac{1}{Cs}}$; $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{RCs + 1}$

(c) Based on your answer in (b), find the impulse response $h(t)$. *6 points.* (See Lathi, Table 4.1, p. 344)

$$H(s) = \frac{1}{RCs + 1} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}$$

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$

(d) Give a formula for the system time constant in terms of R and C. *3 points.*

$\tau = RC$

(see Lathi, Sec. 2.7-2, pp. 216-218, as well as problem 2.1 on this test.)

Problem 2.3 Discrete-Time Signals. *21 points.*

The unilateral z-transform transforms a discrete-time function $x[n]$ into a function $X[z]$ of a complex-variable z as follows:

$$X[z] = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

(a) Using the z-transform definition above, find the z-transform of the discrete-time impulse $\delta[n]$. Please include the region of convergence. *6 points.* (See Lathi, Example 5.2, p. 499.)

The discrete-time impulse is defined as

$$\delta[n] = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

$$X[z] = \sum_{n=0}^{\infty} \delta[n]\, z^{-n} = \sum_{n=0}^{0} z^{-n} = 1 \quad \text{for all } z$$

(b) Using the z-transform definition above, find the z-transform of a causal exponential signal $a^n\, u[n]$ where a is a complex-valued constant. Please include the region of convergence. *6 points.*

(see Lathi, Example 5.1, p. 496.)

$$X[z] = \sum_{n=0}^{\infty} a^n u[n]\, z^{-n} \qquad u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

which can be put under a common exponent n because n is an integer.

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1-\frac{a}{z}} \text{ if } \left|\frac{a}{z}\right| < 1 = \frac{z}{z-a} \text{ if } \underbrace{|z| > |a|}_{\text{ROC}}$$

(c) Using z-transforms, compute the convolution $v[n] = a^n\, u[n] * b^n\, u[n]$ where a and b are constants so that $a \neq b$. Please track the region of convergence in your calculations. *9 points.*

$$v_1[n] = a^n u[n] \longrightarrow V_1[z] = \frac{z}{z-a} \text{ if } |z| > |a|$$

$$v_2[n] = b^n u[n] \longrightarrow V_2[z] = \frac{z}{z-b} \text{ if } |z| > |b|$$

$$v[n] = v_1[n] * v_2[n] \longrightarrow V[z] = V_1[z]\, V_2[z]$$

ROC is $|z| > \max\{|a|, |b|\}$

$$V[z] = \frac{z}{z-a}\,\frac{z}{z-b} \longrightarrow \frac{V[z]}{z} = \frac{z}{(z-a)(z-b)}$$

$$= \frac{\frac{a}{a-b}}{z-a} + \frac{\frac{b}{b-a}}{z-b}$$

$$V[z] = \left(\frac{a}{a-b}\right)\frac{z}{z-a} - \left(\frac{b}{a-b}\right)\frac{z}{z-b} \rightarrow v[n] = \frac{1}{a-b}\left[a^{n+1} - b^{n+1}\right]u[n]$$

Problem 2.4 Discrete-Time Systems. *21 points.*

Consider a causal discrete-time linear time-invariant system with input $x[n]$ and output $y[n]$ being governed by the following difference equation:

$$y[n] = (2 \cos \omega_0)\, y[n-1] - y[n-2] + x[n] - (\cos \omega_0)\, x[n-1]$$

The impulse response is a *causal sinusoid* with constant discrete-time frequency ω_0 in units of rad/sample.

(a) Please state all initial conditions. Please give values for the initial conditions to satisfy the stated system properties. *6 points.*

Start computing output values to reveal the initial conditions in the system:

$$y[0] = (2\cos w_0)\, y[-1] - y[-2] + x[0] - (\cos w_0)\, x[-1]$$

$$y[1] = (2\cos w_0)\, y[0] - y[-1] + x[1] - (\cos w_0)\, x[0]$$

Initial conditions are $y[-1]$, $y[-2]$ and $x[-1]$.

The initial conditions must be zeroed out to satisfy linearity and time-invariance properties.

(b) Find the equation for the transfer function $H[z]$ in the z-domain, including the region of convergence. *9 points.*

Taking the z-transform of both sides of the difference equation with initial conditions being set to zero,

$$Y[z] = (2\cos w_0)\, z^{-1} Y[z] - z^{-2} Y[z] + X[z] - (\cos w_0)\, z^{-1} X[z]$$

$$Y[z]\left(1 - (2\cos w_0)\, z^{-1} + z^{-2}\right) = X[z]\left(1 - (\cos w_0)\, z^{-1}\right)$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{1 - (\cos w_0)\, z^{-1}}{1 - (2\cos w_0)\, z^{-1} + z^{-2}} = \frac{z^2 - (\cos w_0)\, z}{z^2 - (2\cos w_0)\, z + 1}$$

(c) Find the inverse z-transform of the transfer function in part (b) to find the formula for the impulse response $h[n]$ of the system. *6 points.*

From the problem statement, the impulse response is a causal sinusoid.

From Lathi, Table 5.1, p. 498, we can use the z-transform table:

$$h[n] = \cos(w_0 n)\, u[n]$$

Roots of the denominator:

$$\frac{-2\cos w_0 \pm \sqrt{4\cos^2 w_0 - 4}}{2} =$$

$$\cos w_0 \pm \sqrt{\cos^2 w_0 - 1} =$$

$$\cos w_0 \pm j \sin w_0 = e^{\pm j w_0}$$

ROC must lie outside of pole radii for causal inverse: $|z| > 1$

Problem 2.5 Transform Connections. *16 points.*

The unilateral Laplace transform transforms a continuous-time function $x(t)$ into a function $X(s)$ of a complex-variable s as follows:

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st}\, dt$$

The unilateral z-transform transforms a discrete-time function $x[n]$ into a function $X[z]$ of a complex-variable z as follows:

$$X[z] = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Consider the following mapping from the s-plane to the z-plane

$$z = e^{s\,T}$$

where T is a positive constant ($T > 0$) with units in seconds.

(See Lathi, Section 5.8, pp. 560-562)

(a) For the above mapping $z = \exp(s\, T)$, write the z variable in polar form $r \exp(j\, \omega)$ and the s variable in Cartesian form $\sigma + j\, \Omega$. The variable Ω is a frequency in units of rad/s.

$$z = e^{sT}$$

$$r e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r e^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

i. Give the formula for r in terms of σ, Ω and T. *3 points.*

$$r = e^{\sigma T}$$

ii. Give the formula of ω in terms of σ, Ω and T. *3 points.*

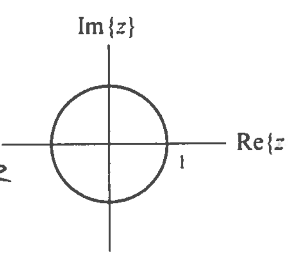
$$\omega = \Omega T$$

(b) To where in the z-plane would the following values of s-plane map?

i. $\text{Re}\{s\} = 0$. Imaginary axis of the s-plane, i.e. $s = j\,\Omega$. *4 points.*

$$z = e^{sT} = e^{j\Omega T}$$

$$\left| e^{j\Omega T} \right| = 1 \longrightarrow \text{on unit circle in z-plane}$$



z-plane with
circle of radius 1

ii. $\text{Re}\{s\} < 0$. Left-hand side of the s-plane. *3 points*

$$z = e^{sT} = e^{\sigma T} e^{j\Omega T}$$

$$\sigma < 0 \longrightarrow 0 \leq r < 1 \longrightarrow \text{inside unit circle in z-plane}$$

iii. $\text{Re}\{s\} > 0$. Right-hand side of the s-plane. *3 points.*

$$z = e^{sT} = e^{\sigma T} e^{j\Omega T}$$

$$\sigma > 0 \longrightarrow r > 1 \longrightarrow \text{outside unit circle in z-plane}$$